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A remark on the definitions of viscosity solutions for the integro-differential equations with Lévy operators

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Abstract

We prove the equivalence of the three different definitions of the viscosity solution for the integro-differential equation with the Lévy operator. The two of the definitions are known in the preceding works of the author and the others, and the last one is new. A construction of a sequence of the approximating test functions to the subsolution (or the supersolution) is indispensable for the proof, and it is done explicitly in the paper.

Résumé

Nous montrons l'équivalence de trois définitions différentes de la solution de viscosité pour les équations intégro-différentielles pour l'opérateur de Lévy. Deux des trois définitions sont connues dans les travaux précédents, et une est nouvelle. Une construction d'une suite des fonctions test est indispensable à la démonstration, et cela est donnée explicitement dans l'article.

Keywords: Integro-differential equation; Lévy operator; Viscosity solution

1. Introduction

In this note, we shall consider the following problem:

$$F\big(x, u, \nabla u, \nabla^2 u\big) - \int_{\mathbf{R^N}} \big[u(x+z) - u(x) - \mathbf{1}_{|z|\leqslant 1}\big\langle \nabla u(x), z\big\rangle\big] q(z)\,dz = 0, \quad x \in \Omega, \tag{1}$$

where $\Omega \subset \mathbf{R^N}$, $F \in C(\Omega \times \mathbf{R} \times \mathbf{R^N} \times \mathbf{S^N})$ is a second-order fully nonlinear elliptic operator, and the Lévy measure $q(z)\,dz$ is a positive Radon measure such that

$$\int_{|z|<1} |z|^2 q(z)\,dz + \int_{|z|\geqslant 1} 1q(z)\,dz < \infty. \tag{2}$$

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The above type of the problem is interested from the view point of the application in the mathematical finances (see Cont and Tankov [7], Sulem and Oksendal [12]). The comparison and the existence results have been studied in some frameworks of the viscosity solutions. However, the equivalence between these notions of viscosity solutions for (1) are not trivial. Here, we would like to give some remarks on the relationships between viscosity solutions defined in different manners.

For an upper (resp. lower) semicontinuous function $u \in USC(\mathbf{R^N})$ (resp. $LSC(\mathbf{R^N})$), we say that $(p, X) \in \mathbf{R^N} \times \mathbf{S^N}$ a subdifferential (resp. superdifferential) of u at x, if for any $\delta > 0$ there exists $\varepsilon > 0$ such that

$$u(x+z) - u(x) \leqslant \langle p, z\rangle + \frac{1}{2}\langle Xz, z\rangle + \delta|z|^2 \quad \forall |z| \leqslant \varepsilon. \tag{3}$$

(resp.

$$u(x+z) - u(x) \geqslant \langle p, z\rangle + \frac{1}{2}\langle Xz, z\rangle - \delta|z|^2 \quad \forall |z| \leqslant \varepsilon.) \tag{4}$$

We denote the set of all subdifferentials (resp. superdifferentials) of u at x $J^{2,+}_{\mathbf{R^N}} u(x)$ (resp. $J^{2,-}_{\mathbf{R^N}} u(x)$). As is well-known (see Crandall, Ishii and Lions [8]), if (p, X) is a subdifferential (resp. superdifferential) of u at x, then there exists $\phi \in C^2(\mathbf{R^N})$ such that $u(x) = \phi(x)$, $u - \phi$ takes a global maximum (resp. minimum) at x, and for any $\delta > 0$ there exists $\varepsilon > 0$ such that

$$u(x+z) - u(x) \leqslant \phi(x+z) - \phi(x) \leqslant \big\langle \nabla\phi(x), z\big\rangle + \frac{1}{2}\big\langle \nabla^2\phi(x)z, z\big\rangle + \delta|z|^2 \quad \forall |z| \leqslant \varepsilon. \tag{5}$$

(resp.

$$u(x+z) - u(x) \geqslant \phi(x+z) - \phi(x) \geqslant \big\langle \nabla\phi(x), z\big\rangle + \frac{1}{2}\big\langle \nabla^2\phi(x)z, z\big\rangle - \delta|z|^2 \quad \forall |z| \leqslant \varepsilon.) \tag{6}$$

In Arisawa [1–3], the following definition of the viscosity solutions for (1) was introduced.

Definition A. Let $u \in USC(\mathbf{R^N})$ (resp. $v \in LSC(\mathbf{R^N})$). We say that u (resp. v) is a viscosity subsolution (resp. supersolution) of (1), if for any $\hat{x} \in \Omega$, any $(p, X) \in J^{2,+}_{\mathbf{R^N}} u(\hat{x})$ (resp. $\in J^{2,-}_{\mathbf{R^N}} v(\hat{x})$), and any pair of numbers (ε, δ) satisfying (3) (resp. (4)), the following holds:

$$F\big(\hat{x}, u(\hat{x}), p, X\big) - \int\limits_{|z|<\varepsilon} \frac{1}{2}\big\langle (X + 2\delta I)z, z\big\rangle q(z)\,dz - \int\limits_{|z|\geqslant\varepsilon} \big[u(\hat{x}+z) - u(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\langle z, p\rangle\big] q(z)\,dz \leqslant 0. \tag{7}$$

(resp.

$$F\big(\hat{x}, v(\hat{x}), p, X\big) - \int\limits_{|z|<\varepsilon} \frac{1}{2}\big\langle (X - 2\delta I)z, z\big\rangle q(z)\,dz - \int\limits_{|z|\geqslant\varepsilon} \big[v(\hat{x}+z) - v(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\langle z, p\rangle\big] q(z)\,dz \geqslant 0.) \tag{8}$$

If u is both a viscosity subsolution and a viscosity supersolution, it is called a viscosity solution.

We can rephrase Definition A by using the test functions in (5) (resp. (6)) as follows:

Definition A′. Let $u \in USC(\mathbf{R^N})$ (resp. $v \in LSC(\mathbf{R^N})$). We say that u (resp. v) is a viscosity subsolution (resp. supersolution) of (1), if for any $\hat{x} \in \Omega$ and for any $\phi \in C^2(\mathbf{R^N})$ such that $u(\hat{x}) = \phi(\hat{x})$ and $u - \phi$ takes a global maximum (resp. minimum) at $\hat{x}$, and for any pair of numbers (ε, δ) satisfying (5) (resp. (6)), the following holds:

$$F\big(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})\big) - \int\limits_{|z|<\varepsilon} \frac{1}{2}\big\langle \big(\nabla^2\phi(\hat{x}) + 2\delta I\big)z, z\big\rangle q(z)\,dz$$
$$- \int\limits_{|z|\geqslant\varepsilon} \big[u(\hat{x}+z) - u(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\big\langle z, \nabla\phi(\hat{x})\big\rangle\big] q(z)\,dz \leqslant 0. \tag{9}$$

(resp.

$$F\big(\hat{x}, v(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})\big) - \int_{|z|<\varepsilon} \frac{1}{2}\big\langle\big(\nabla^2\phi(\hat{x}) - 2\delta I\big)z, z\big\rangle q(z)\,dz$$
$$- \int_{|z|\geqslant\varepsilon} \big[v(\hat{x}+z) - v(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\big\langle z, \nabla\phi(\hat{x})\big\rangle\big]q(z)\,dz \geqslant 0.) \tag{10}$$

If u is both a viscosity subsolution and a viscosity supersolution, it is called a viscosity solution.

We observe that the "global" maximality (resp. minimality) of $u - \phi$ at $\hat{x}$ in Definition A′ can be replaced by the "local" maximality (resp. minimality), without changing any meaning of the definition. It is also clear that Definitions A and A′ are equivalent. Next, we state the following definition of the viscosity solution in Barles, Buckdahn and Pardoux [4], Jacobsen and Karlsen [10], Barles and Imbert [5].

Definition B. Let $u \in USC(\mathbf{R^N})$ (resp. $v \in LSC(\mathbf{R^N})$). We say that u (resp. v) is a viscosity subsolution (resp. supersolution) of (1), if for any $\hat{x} \in \Omega$ and for any $\phi \in C^2(\mathbf{R^N})$ such that $u(\hat{x}) = \phi(\hat{x})$ and $u - \phi$ takes a global maximum (resp. minimum) at $\hat{x}$,

$$F\big(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})\big) - \int_{z\in\mathbf{R^N}} \big[\phi(\hat{x}+z) - \phi(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\big\langle z, \nabla\phi(\hat{x})\big\rangle\big]q(z)\,dz \leqslant 0. \tag{11}$$

(resp.

$$F\big(\hat{x}, v(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})\big) - \int_{z\in\mathbf{R^N}} \big[\phi(\hat{x}+z) - \phi(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\big\langle z, \nabla\phi(\hat{x})\big\rangle\big]q(z)\,dz \geqslant 0.) \tag{12}$$

If u is both a viscosity subsolution and a viscosity supersolution, it is called a viscosity solution.

Remark 1.1. In the above cited works, Definition B was stated to be equivalent to the following definition. See [10] for the proof.

Definition B′. Let $u \in USC(\mathbf{R^N})$ (resp. $v \in LSC(\mathbf{R^N})$). We say that u (resp. v) is a viscosity subsolution (resp. supersolution) of (1), if for any $\hat{x} \in \Omega$ and for any $\phi \in C^2(\mathbf{R^N})$ such that $u(\hat{x}) = \phi(\hat{x})$ and $u - \phi$ takes a global maximum (resp. minimum) at $\hat{x}$, and for any $\varepsilon > 0$,

$$F\big(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})\big) - \int_{|z|<\varepsilon} \big[\phi(\hat{x}+z) - \phi(\hat{x}) - \big\langle z, \nabla\phi(\hat{x})\big\rangle\big]q(z)\,dz$$
$$- \int_{|z|\geqslant\varepsilon} \big[u(\hat{x}+z) - u(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\big\langle z, \nabla\phi(\hat{x})\big\rangle\big]q(z)\,dz \leqslant 0. \tag{13}$$

(resp.

$$F\big(\hat{x}, v(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})\big) - \int_{|z|<\varepsilon} \big[\phi(\hat{x}+z) - \phi(\hat{x}) - \big\langle z, \nabla\phi(\hat{x})\big\rangle\big]q(z)\,dz$$
$$- \int_{|z|\geqslant\varepsilon} \big[v(\hat{x}+z) - v(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\big\langle z, \nabla\phi(\hat{x})\big\rangle\big]q(z)\,dz \geqslant 0.) \tag{14}$$

If u is both a viscosity subsolution and a viscosity supersolution, it is called a viscosity solution.

Here, we shall consider Definition B.

In this paper, thirdly we are interested in the following definition of the viscosity solution, which seems to be stronger than others at a first glance.

Definition C. Let $u \in USC(\mathbf{R^N})$ (resp. $v \in LSC(\mathbf{R^N})$). We say that u (resp. v) is a viscosity subsolution (resp. supersolution) of (1), if for any $\hat{x} \in \Omega$ and for any $\phi \in C^2(\mathbf{R^N})$ such that $u(\hat{x}) = \phi(\hat{x})$ and $u - \phi$ takes a global maximum (resp. minimum) at $\hat{x}$, the function $h(z) = u(\hat{x}+z) - u(\hat{x}) - \langle z, \nabla\phi(\hat{x})\rangle \in L^1(\mathbf{R^N}, q(z)\,dz)$ (resp. $h(z) = v(\hat{x}+z) - v(\hat{x}) - \langle z, \nabla\phi(\hat{x})\rangle \in L^1(\mathbf{R^N}, q(z)\,dz)$) and

$$F\big(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})\big) - \int\limits_{z\in\mathbf{R^N}} \big[u(\hat{x}+z) - u(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\big\langle z, \nabla\phi(\hat{x})\big\rangle\big] q(z)\,dz \leqslant 0. \tag{15}$$

(resp.

$$F\big(\hat{x}, v(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})\big) - \int\limits_{z\in\mathbf{R^N}} \big[v(\hat{x}+z) - v(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\big\langle z, \nabla\phi(\hat{x})\big\rangle\big] q(z)\,dz \geqslant 0.) \tag{16}$$

If u is both a viscosity subsolution and a viscosity supersolution, it is called a viscosity solution.

We state the following result on the relationships between Definitions A, B and C.

Theorem 1.1. *Definitions A, B, and C are equivalent.*

In the following sections, we shall prove the above statement.

2. Some lemmas

Let $u(x)$ be an upper semi-continuous function. Assume that there exists $\phi(x) \in C^2(\mathbf{R^N})$, such that $u - \phi$ takes a global maximum at a point $\hat{x} \in \mathbf{R^N}$ and $u(\hat{x}) = \phi(\hat{x})$. First, we need to construct a sequence of approximating test functions. There are some ways to do that, and the direction of the following construction owes to P.-L. Lions [11]. See also Lemma 2.2 and Remark 2.2 in below for other constructions.

Lemma 2.1. *Let $u(x) \in USC(\mathbf{R^N})$. Assume that there exists $\phi(x) \in C^2(\mathbf{R^N})$, such that $u - \phi$ takes a global maximum at a point $\hat{x} \in \mathbf{R^N}$ and $u(\hat{x}) = \phi(\hat{x})$. Then, there exists a monotone decreasing sequence of functions $\phi_n(x) \in C^2(\mathbf{R^N})$ such that $u - \phi_n$ takes a global maximum at a point $\hat{x} \in \mathbf{R^N}$, $u(\hat{x}) = \phi_n(\hat{x})$, $\nabla\phi_n(\hat{x}) = \nabla\phi(\hat{x})$, $\nabla^2\phi_n(\hat{x}) = \nabla^2\phi(\hat{x})$, and*

$$u(x) \leqslant \phi_n(x) \leqslant \phi(x) \quad \forall x \in \mathbf{R^N},\ \forall n, \qquad \phi_n(x) \downarrow u(x)\ \forall x \in \mathbf{R^N} \text{ as } n \to \infty.$$

Proof. It is enough to prove the case that u is continuous. (If $u \in USC(\mathbf{R^N})$ is not continuous, we approximate it by using the sup-convolutions.) We may also assume that $u - \phi$ takes the strict maximum at $\hat{x}$, for if not we may add a small positive quadratic function to ϕ. In the following, we construct ϕ_n inductively.

Step 1. First, we shall choose $\phi_1(x) \in C^2(\mathbf{R^N})$. Let $B(\hat{x})$ be an open ball centered at $\hat{x}$ with radius $s > 0$. There exists a sequence of functions $\psi_m(x) \in C^2(\mathbf{R^N})$ $(m \in \mathbf{N})$ such that

$$\lim_{m\to\infty} \psi_m(x) = u(x) \quad \text{uniformly in } B(\hat{x}).$$

Put $\overline{\psi}_m(x) = \psi_m(x) + |u - \psi_m|_{L^\infty(B(\hat{x}))}$. Then,

$$\overline{\psi}_m(x) \geqslant u(x) \quad \forall x \in B(\hat{x}), \qquad \lim_{m\to\infty} \overline{\psi}_m(x) = u(x) \quad \text{uniformly in } B(\hat{x}).$$

Since $u - \phi$ takes the strict maximum at $\hat{x}$, for any $r > 0$ such that $r < s$ there exists $\sigma(r) > 0$ such that

$$\min_{x\in B(\hat{x}),\,|x-\hat{x}|\geqslant r} (\phi - u)(x) = \sigma(r) > 0.$$

Now, take $\chi_r(x) \in C^2(B(\hat{x}))$ such that

$$\chi_r(x) = \begin{cases} 1 & \text{if } |x-\hat{x}| \leqslant \frac{r}{2}; \\ 0 & \text{if } r \leqslant |x-\hat{x}| \leqslant s, \end{cases}$$

and $0 \leqslant \chi_r(x) \leqslant 1$ for any $x \in B(\hat{x})$. Define

$$\overline{\phi}_m(x) = \chi_r(x)\phi(x) + \big(1 - \chi_r(x)\big)\overline{\psi}_m(x) \quad \forall x \in B(\hat{x}).$$

Clearly, $\overline{\phi}_m(x) \in C^2(B(\hat{x}))$,

$$\overline{\phi}_m(x) \geqslant u(x) \quad \forall x \in B(\hat{x}),\ \forall m \in \mathbf{N}.$$

Moreover,

$$\overline{\phi}_m(x) = \phi(x) \quad \text{if } |x - \hat{x}| \leqslant \frac{r}{2},$$

and since $\phi(x) - \overline{\phi}_m(x) = (1 - \chi_r(x))(\phi(x) - \overline{\psi}_m(x))$,

$$\begin{aligned}\phi(x) - \overline{\psi}_m(x) &= \phi(x) - u(x) + u(x) - \overline{\psi}_m(x)\\ &\geqslant \sigma\left(\frac{r}{2}\right) + u(x) - \overline{\psi}_m(x) > 0 \quad \text{if } \frac{r}{2} \leqslant |x - \hat{x}| \leqslant s,\ \text{for } m \text{ large enough,}\end{aligned}$$

we have

$$\phi(x) - \overline{\phi}_m(x) \geqslant 0 \quad \text{if } \frac{r}{2} \leqslant |x - \hat{x}| \leqslant s,$$

for m large enough. Since

$$\big|\overline{\phi}_m(x) - u(x)\big| \leqslant \big|\phi(x) - u(x)\big|, \quad |x - \hat{x}| < r, \quad \phi(\hat{x}) = u(\hat{x}),$$

and

$$\lim_{m\to\infty}\big|\overline{\phi}_m(x) - u(x)\big| = 0 \quad \text{uniformly in } r \leqslant |x - \hat{x}| \leqslant s,$$

by taking $r > 0$ small enough, and then m large enough, we get a function $\overline{\phi}_m(x) \in C^2(B(\hat{x}))$ such that

$$u(x) \leqslant \overline{\phi}_m(x) \leqslant \phi(x), \qquad \big|\overline{\phi}_m(x) - u(x)\big| \leqslant \frac{1}{s} \quad \text{for } |x - \hat{x}| \leqslant s.$$

We can extend this $\overline{\phi}_m(x)$ to the whole space so that the extended function $\overline{\phi}_m(x) \in C^2(\mathbf{R^N})$, satisfies

$$u(x) \leqslant \overline{\phi}_m(x) \leqslant \phi(x), \quad x \in \mathbf{R^N},$$

and $u - \overline{\phi}_m(x)$ takes a global strict maximum at $\hat{x}$. Put $\phi_1(x) = \overline{\phi}_m(x)$, and remark that $\phi_1(x)$ is first constructed on $B(\hat{x}) = \{x \in \mathbf{R^N} \mid |x - \hat{x}| < s\}$, and

$$\big|\phi_1(x) - u(x)\big| \leqslant \frac{1}{s} \quad \text{for } |x - \hat{x}| \leqslant s.$$

Step 2. We repeat the above argument on $B_2(\hat{x}) = \{x \in \mathbf{R^N} \mid |x - \hat{x}| < 2s\}$, by using $\phi_1(x)$ instead of $\phi(x)$. Thus, we get $\phi_2(x) \in C^2(\mathbf{R^N})$ such that

$$u(x) \leqslant \phi_2(x) \leqslant \phi_1(x), \quad x \in \mathbf{R^N},$$

$$\big|\phi_2(x) - u(x)\big| \leqslant \frac{1}{2s} \quad \text{for } |x - \hat{x}| \leqslant 2s,$$

and that $u - \phi_2(x)$ takes a global strict maximum at $\hat{x}$. Remark that $\phi_2(x)$ is first constructed on $B_2(\hat{x}) = \{x \in \mathbf{R^N} \mid |x - \hat{x}| < 2s\}$.

Step 3. The above procedure shows inductively the existence of $\phi_n(x) \in C^2(\mathbf{R^N})$ such that

$$u(x) \leqslant \phi_n(x) \leqslant \phi_{n-1}(x), \quad x \in \mathbf{R^N},$$

$$\big|\phi_n(x) - u(x)\big| \leqslant \frac{1}{ns} \quad \text{for } |x - \hat{x}| \leqslant ns,$$

and $u - \phi_n(x)$ takes a global strict maximum at $\hat{x}$. Moreover, the above construction shows that for each n there exists a small neighborhood $\{x \mid |x - \hat{x}| < r(n)\}$ such that $\phi_n = \phi$. Therefore, it is clear that ϕ_n ($n \in \mathbf{N}$) satisfies the claims of the lemma. □

Remark 2.1. The construction of the sequence of the approximating test functions for the supersolution can be done similarly.

To prove the equivalence of the definitions, the property $\phi_n(x) \leqslant \phi(x)$ $(n \in \mathbf{N})$ is not necessary. The following is another type of the construction.

Lemma 2.2. *Let $u(x) \in USC(\mathbf{R^N})$. Assume that there exists $\phi(x) \in C^2(\mathbf{R^N})$, such that $u - \phi$ takes a global maximum at a point $\hat{x} \in \mathbf{R^N}$ and $u(\hat{x}) = \phi(\hat{x})$. Then, there exists a monotone decreasing sequence of functions $\phi_n(x) \in C^2(\mathbf{R^N})$ such that $u - \phi_n$ takes the global maximum at $\hat{x}$, $u(\hat{x}) = \phi_n(\hat{x})$, $\nabla\phi(\hat{x}) = \nabla\phi_n(\hat{x})$, and*

$$\phi_n(x) \downarrow u(x) \quad \forall x \in \mathbf{R^N}, \text{ as } n \to \infty.$$

Proof. We may assume that $\hat{x} = 0$, $u(\hat{x}) = \phi(\hat{x}) = 0$, $\nabla\phi(\hat{x}) = 0$, without any loss of the generality. Now, since $\phi \in C^2(\mathbf{R^N})$, we can take $M_n = \sup_{|x| \leqslant n^{-1}} |\nabla^2\phi(0)|$ for any $n \in \mathbf{N}$. Put $\phi_n^0(x) = 2M_n|x|^2$ in $\{x \in \mathbf{R^N} \mid |x| \leqslant n^{-1}\}$, and remark that $\phi_n^0 \geqslant \phi$ in the domain of the definition. Remark also that $M_n > 0$ $(n \in \mathbf{N})$ is monotone decreasing. We can take $\phi_n \in C^2(\mathbf{R^N})$ such that

$$\phi_n(x) = \phi_n^0(x) \quad \text{for } |x| \leqslant \frac{1}{2n}; \qquad u(x) \leqslant \phi_n(x) \leqslant u(x) + n^{-1} \quad \text{for } |x| \geqslant \frac{2}{n},$$
$$\phi_{n+1}(x) \leqslant \phi_n(x) \quad \text{on } \mathbf{R^N}, \text{ for } \forall n \in \mathbf{N},$$

and that $\phi_n - u$ takes its global maximum at 0 for any $n \in \mathbf{N}$. The sequence of functions $\{\phi_n\}$ $(n \in \mathbf{R^N})$ satisfies the claims in the lemma. □

Remark 2.2. The idea of the above construction comes from a result in Evans [9]. The convergence of the second-order derivatives of the test functions:

$$\nabla^2\phi_n(\hat{x}) \downarrow \nabla^2\phi(\hat{x}) \quad \text{as } n \to \infty, \tag{17}$$

does not hold in general. Yet, we can use this lemma for the proof of Theorem 1.1, when the PDE operator F is first-order, i.e. $F(x, u, \nabla u, \nabla^2 u) = F(x, u, \nabla u)$. It is possible to improve the construction so that ϕ_n satisfies (17). However, the way is more redundant than Lemma 2.1, and we do not write it here.

We shall use the following well-known elementary theorem of the monotone convergence of Beppo-Levi, too.

Lemma 2.3. *(Beppo-Levi, see H. Brezis [6].) Let $f_n(x)$ $(n \in \mathbf{N})$ be a sequence of increasing functions in $L^1(\mathcal{O}, d\mu(x))$ $(\mathcal{O} \subset \mathbf{R^N})$, such that $\sup_n \int_O f_n \, d\mu(x) < \infty$. Then, $f_n(x)$ converges almost everywhere in $\mathcal{O}$ to a function $f(x)$. Moreover $f(x) \in L^1$ and $\|f_n - f\|_{L^1} \to 0$ as $n \to \infty$.*

3. Proof of the main result

We divide the proof of Theorem 1.1 into three steps.

Step 1. We shall first show the following:

Lemma 3.1. (i) *If u is a viscosity subsolution of* (1) *in the sense of Definition B, then u is a viscosity subsolution of* (1) *in the sense of Definition C.*

(ii) *If u is a viscosity subsolution of* (1) *in the sense of Definition C, then u is a viscosity subsolution of* (1) *in the sense of Definition B.*

Proof. (i) Let u be a viscosity subsolution of (1) in the sense of Definition B. Assume that there exists $\phi \in C^2(\mathbf{R^N})$ such that $u - \phi$ takes a global maximum at $\hat{x} \in \Omega$, and $u(\hat{x}) = \phi(\hat{x})$. Then from Lemma 2.1, there exists a sequence of functions $\phi_n \in C^2(\mathbf{R^N})$ $(n \in \mathbf{N})$ having the properties in the lemma. Since $u - \phi_n$ $(n \in \mathbf{N})$ takes a global maximum at $\hat{x}$, from Definition B

$$F\big(\hat{x}, u(\hat{x}), \nabla\phi_n(\hat{x}), \nabla^2\phi_n(\hat{x})\big) - \int\limits_{z\in\mathbf{R^N}} \big[\phi_n(\hat{x}+z) - \phi_n(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\langle z, \nabla\phi_n(\hat{x})\rangle\big] q(z)\,dz \leqslant 0 \quad \forall n. \tag{18}$$

Put

$$h_n(z) = \phi_n(\hat{x}+z) - \phi_n(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\langle z, \nabla\phi_n(\hat{x})\rangle \quad \forall n.$$

From Lemma 2.1, $\phi_n(\hat{x}) = u(x)$, $\nabla\phi_n(\hat{x}) = \nabla\phi(\hat{x})$, ϕ_n is monotone decreasing as $n \to \infty$. Thus, $h_n(z)$ is monotone decreasing as $n \to \infty$, too, and

$$\lim_{n\to\infty} h_n(z) = u(\hat{x}+z) - u(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\langle z, \nabla\phi(\hat{x})\rangle.$$

From Lemma 2.3 (Beppo-Levi) we see that

$$u(\hat{x}+z) - u(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\langle z, \nabla\phi(\hat{x})\rangle \in L^1\big(\mathbf{R^N}, q(z)\,dz\big).$$

Therefore, by letting $n \to \infty$ in (18), since $\nabla\phi_n(\hat{x}) = \nabla\phi(\hat{x})$, $\nabla^2\phi_n(\hat{x}) = \nabla^2\phi(\hat{x})$,

$$F\big(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})\big) - \int\limits_{z\in\mathbf{R^N}} \big[u(\hat{x}+z) - u(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\langle z, \nabla\phi(\hat{x})\rangle\big] q(z)\,dz \leqslant 0$$

holds. Hence, u is the viscosity subsolution in the sense of Definition C.

(ii) Let u be a viscosity subsolution of (1) in the sense of Definition C. Assume that there exists $\phi \in C^2(\mathbf{R^N})$ such that $u - \phi$ takes a global maximum at $\hat{x} \in \Omega$, and $u(\hat{x}) = \phi(\hat{x})$. From Definition C,

$$F\big(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})\big) - \int\limits_{z\in\mathbf{R^N}} \big[u(\hat{x}+z) - u(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\langle z, \nabla\phi((\hat{x})\rangle\big] q(z)\,dz \leqslant 0.$$

Since $u(\hat{x}+z) \leqslant \phi(\hat{x}+z)$ for any $z \in \mathbf{R^N}$, it is clear that the above leads

$$F\big(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})\big) - \int\limits_{z\in\mathbf{R^N}} \big[\phi(\hat{x}+z) - \phi(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\langle z, \nabla\phi(\hat{x})\rangle\big] q(z)\,dz \leqslant 0.$$

Therefore, u is the viscosity subsolution in the sense of Definition B. □

Remark 3.1. If F is the first-order Hamiltonian, the approximating sequence ϕ_n ($n \in \mathbf{N}$) in Lemma 2.2 serves to prove the claim in Lemma 3.1.

Step 2. Next, we shall prove the following.

Lemma 3.2. (i) *If u is a viscosity subsolution of* (1) *in the sense of Definition A, then u is a viscosity subsolution of* (1) *in the sense of Definition B.*

(ii) *If u is a viscosity subsolution of* (1) *in the sense of Definition C, then u is a viscosity subsolution of* (1) *in the sense of Definition A.*

Proof. (i) Let u be a viscosity subsolution of (1) in the sense of Definition A. Remark that Definition A is equivalent to Definition A′. Assume that there exists $\phi \in C^2(\mathbf{R^N})$ such that $u - \phi$ takes a global maximum at $\hat{x} \in \Omega$, and $u(\hat{x}) = \phi(\hat{x})$. Then, for any pair of numbers (ε, δ) such that (5) holds,

$$F\big(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})\big) - \int\limits_{|z|<\varepsilon} \frac{1}{2}\big\langle\big(\nabla^2\phi(\hat{x}) + 2\delta I\big)z, z\big\rangle q(z)\,dz$$
$$- \int\limits_{|z|\geqslant\varepsilon} \big[u(\hat{x}+z) - u(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\langle z, \nabla\phi(\hat{x})\rangle\big] q(z)\,dz \leqslant 0.$$

Then, since $u(\hat{x}+z) \leqslant \phi(\hat{x}+z)$ for any $z \in \mathbf{R^N}$,

$$F\big(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})\big) - \int\limits_{|z|<\varepsilon} \frac{1}{2}\big\langle\big(\nabla^2\phi(\hat{x}) + 2\delta I\big)z, z\big\rangle q(z)\, dz$$
$$- \int\limits_{|z|\geqslant\varepsilon} \big[\phi(\hat{x}+z) - \phi(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\big\langle z, \nabla\phi(\hat{x})\big\rangle\big] q(z)\, dz \leqslant 0.$$

By tending $\varepsilon \to 0$, this shows that u is the viscosity subsolution in the sense of Definition B.

(ii) Let u be a viscosity subsolution of (1) in the sense of Definition C. Assume that there exists $\phi \in C^2(\mathbf{R^N})$ such that $u - \phi$ takes a global maximum at $\hat{x} \in \Omega$, and $u(\hat{x}) = \phi(\hat{x})$. We have:

$$F\big(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})\big) - \int\limits_{z\in\mathbf{R^N}} \big[u(\hat{x}+z) - u(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\big\langle z, \nabla\phi(\hat{x})\big\rangle\big] q(z)\, dz \leqslant 0.$$

Since

$$u(\hat{x}+z) - u(\hat{x}) - \big\langle z, \nabla\phi(\hat{x})\big\rangle \leqslant \phi(\hat{x}+z) - \phi(\hat{x}) - \big\langle z, \nabla\phi(\hat{x})\big\rangle$$
$$\leqslant \frac{1}{2}\big\langle \nabla^2\phi(\hat{x})z, z\big\rangle + \delta|z|^2, \quad |z| \leqslant \varepsilon,$$

we have:

$$F\big(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})\big) - \int\limits_{|z|<\varepsilon} \frac{1}{2}\big\langle\big(\nabla^2\phi(\hat{x}) + 2\delta I\big)z, z\big\rangle q(z)\, dz$$
$$- \int\limits_{|z|\geqslant\varepsilon} \big[u(\hat{x}+z) - u(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\big\langle z, \nabla\phi(\hat{x})\big\rangle\big] q(z)\, dz \leqslant 0.$$

That is, u is a viscosity subsolution of (1) in the sense of Definition A. □

Step 3. For the viscosity supersolutions, the similar claims to those in Lemmas 3.1 and 3.2 hold, too. Thus, it is clear that Definitions A, B and C are equivalent. Theorem 1.1 is proved.

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